Technical Comments

Comment on "Effect of Higher Order Terms in Certain Nonlinear Finite Element Models"

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ODEN et al.[1] conclude that convergence will be slowed if terms which characterize nonlinear behavior are approximated by a simpler polynomial than that used for the conventional stiffness matrix. The present Comment offers numerical evidence pertinent to this theoretical conclusion.

The problem chosen is of the type used in Ref. 1. A cantilever beam is modeled by ten elements, and at its free end carries an axial compressive load P and a transverse load $Q = 0.001$. Properties are chosen such that the Euler buckling load is $P_{cr} = 1.000$. The solution method is based on convected coordinates, as clearly detailed for plate problems by Murray and Wilson.[2] Thus the technique is "pure" Newton-Raphson iteration; under-relaxation, convergence-acceleration schemes, etc., were not used. Iterative cycling at a given load level was terminated by a convergence test on displacements. The next load level was then applied, and iteration begun again, starting with the displacements obtained at the previous load level.

The conventional stiffness matrix of each element (in its convected coordinate system) was based on a cubic polynomial. Two forms of 'geometric' or 'initial stress' stiffness matrix were considered.[3] The first, $[k_{GC}]$, was based on the cubic polynomial, and the second, $[k_{GL}]$, was based on a linear polynomial. In other words, with reference to Eqs. (3) of Ref. 1, $[k_{GC}]$ was based upon the lateral displacement $v(x) = \phi(x)\mathbf{v}$ and $[k_{GL}]$ upon $v(x) = \psi(x)\mathbf{v}$.

Let d/L be the ratio of lateral tip displacement to beam length. With $P = 0.99$, d/L converged to 0.0606 in 9 cycles when $[k_{GL}]$ was used, and to 0.0701 in 15 cycles when $[k_{GC}]$ was used. With the geometric stiffness matrix omitted altogether, d/L reached 0.0257 at 40 cycles but had not converged. The correct result is $d/L = 0.0811$, which is some 100 times the value produced by lateral load Q acting alone.

Table 1 lists tip rotations obtained by progressive increases in load level, with a milder convergence tolerance than used in the preceding paragraph.

Table 1 Tip rotation in radians, with cycles (in parentheses) needed for convergence in five-step loading program.

Geometric stiffness matrix	$P = 0.99$	$P = 1.05$	$P = 1.17$	$P = 1.41$	$P = 1.89$
none	0.04 ...[a]	0.35 ...[a]	1.09 (18)	1.60 (6)	2.10 (3)
$[k_{GC}]$	0.12 (7)	0.61 (22)	1.08 (12)	1.50 (8)	1.98 (13)
$[k_{GL}]$	0.10 (7)	0.62 (8)	1.10 (6)	1.61 (4)	2.11 (4)
theory[4]	...	0.62	1.09	1.59	2.10

[a] Not converged. Rotation at 40 cycles is given.

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Rather than prescribing P and calculating its displacement, one may prescribe an axial displacement and calculate the corresponding P. In a loading program of six prescribed displacement levels, convergence to the correct P was roughly three times faster in each level when the geometric stiffness matrix was omitted altogether.

The foregoing numerical results conflict with the conclusions in Ref. 1. One must therefore ask whether these conclusions are not applicable to a solution based on convected coordinates, or are otherwise of a more restrictive nature than has been made clear.

References

[1] Oden, J. T., Akay, H. U., and Johnson, C. P., "Effect of Higher Order Terms in Certain Nonlinear Finite Element Models," *AIAA Journal*, Vol. 11, No. 11, Nov. 1973, pp. 1589–1590.

[2] Murray, D. W. and Wilson, E. L., "Finite Element Large Deflection Analysis of Plates," *Proceedings of the ASCE, Journal of the Engineering Mechanics Division*, Vol. 95, No. EMI, Feb. 1969, pp. 143–165.

[3] Martin, H. C., "On the Derivation of Stiffness Matrices for the Analysis of Large Deflection and Stability Problems," *Proceedings of the 1st Conference on Matrix Methods in Structural Mechanics*, AFFDL-TR-66-80, Air Force Flight Dynamics Lab., Wright-Patterson Air Force Base, Ohio, Oct. 1965, pp. 697–716.

[4] Timoshenko, S. P. and Gere, J. M., *Theory of Elastic Stability*, 2nd ed., McGraw-Hill, New York, 1961, pp. 76–82.

Reply by Authors to R. D. Cook

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THE authors wish to thank Cook for his interesting comments. We hasten to point out, however, the following:

1) The numerical results that Cook quotes have nothing to do with the subject paper; as is indicated in the opening statement of the model problem, we consider only the behavior of the structure under loads $P < P_{cr}$.

2) Even so, the results obtained by Cook can be rather easily explained, as we shall indicate below.

3) More importantly, Cook's conclusions may be misleading; for structures which are unstable at the critical load, entirely different results than those he reports could be obtained. Indeed, the particular problem Cook describes has a special property which promotes the rapid convergence mentioned in his Comment.

Concerning the first point, a check of our analysis still reveals that we assume a locally quadratic, monotone behavior in the potential energy function. That is to say, our analysis holds so long as a second variation of the energy is positive-definite. Stated in still another way, the operators entering into the nonlinear equations are strongly monotone. This is precisely why we limited our analysis to large deformations in which $P < P_{cr}$.

Now, turning to our second point, suppose we remove the assumption that $P < P_{cr}$. Then the question arises as to whether or not the structure is stable at the critical load. The sample problem picked by Cook is the classical Euler problem which is known to be stable at and beyond the critical load. This

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